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EEM/EEE314 Automatic Control Systems

Exam-style questions with solutions

Part 1: Electrical systems

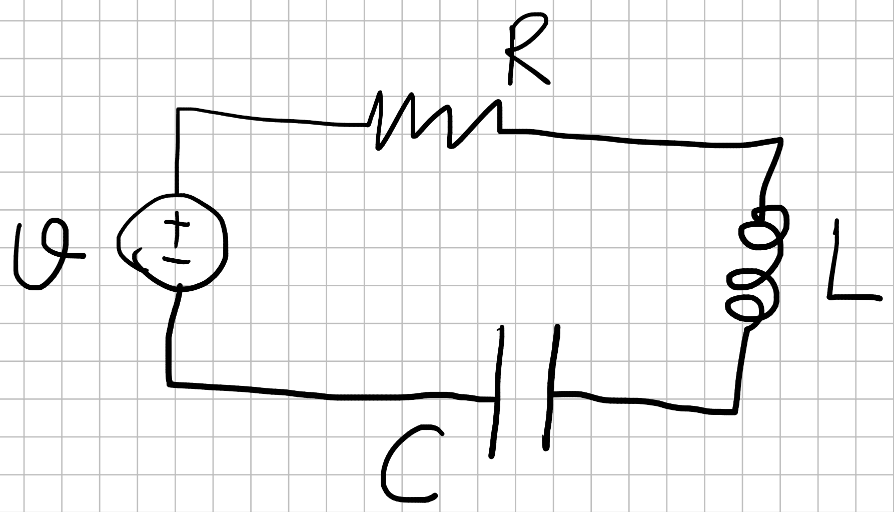
Abbreviations:

KVL: Kirchhoff's voltage law

KCL: Kirchhoff's current law

LHS: left-hand side (of the equation)

Question 1: Consider the schematic of an electrical circuit depicted below, consisting of a resistor, an inductor, and a capacitor (with parameters R, L, and C, respectively). The charge accumulated on the capacitor is denoted as q(t). An external voltage is applied to the circuit, denoted as v(t). Find the differential equation model of the system relating v(t) and q(t).



Solution:

Denoting the voltage drops across the circuit elements as V_R, V_L, and V_c, respectively, and invoking KVL, we write:

$$v = V_R + V_L + V_C$$

Denoting the current flowing through the circuit as $i(t)$, and using linear models of the circuit elements, we write:

$$V_R = R \cdot i \qquad V_L = L \cdot \dot{i}$$

$$V_C = \frac{1}{C} q$$

Note that, for the capacitor, the following are true:

1. $q = \int i \, dt$ (accumulation of charge)

(or, equivalently: $i = \dot{q}$)

2. voltage drop across the capacitor:

$$V_C = \frac{1}{C} q = \frac{1}{C} \int i \, dt$$

(or, equivalently: $i = C \cdot \dot{V}_C$)

The question asks us to relate v and q, thus we need to rewrite V_R and V_L in terms of q (using $i = \dot{q}$):

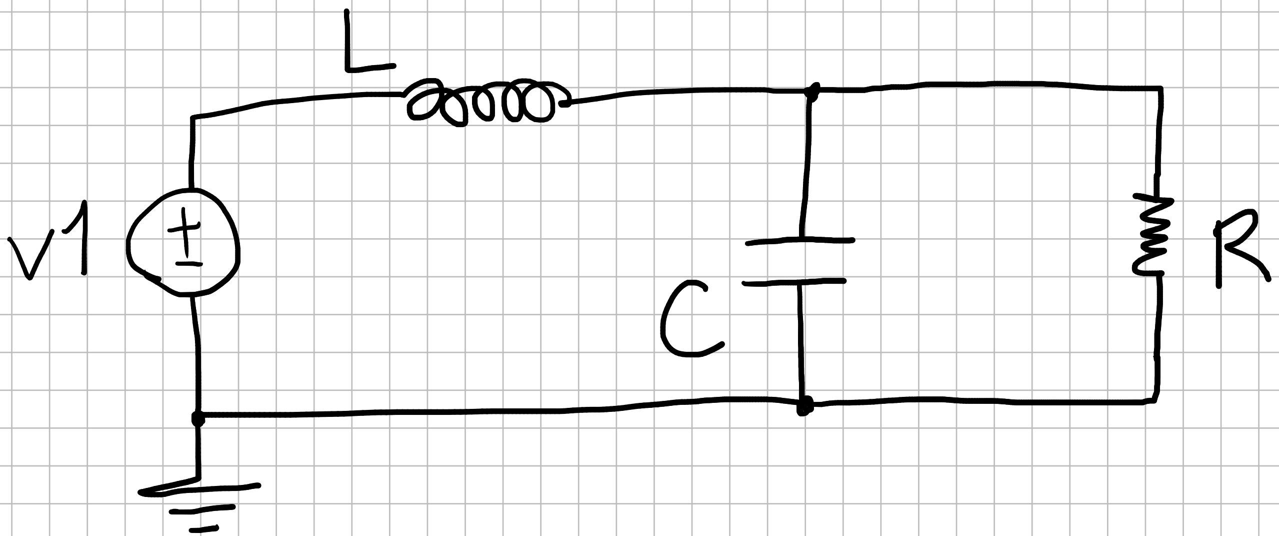
$$V_R = R \cdot i = R \cdot \dot{q}$$

$$V_L = L \cdot \dot{i} = L \cdot \ddot{q}$$

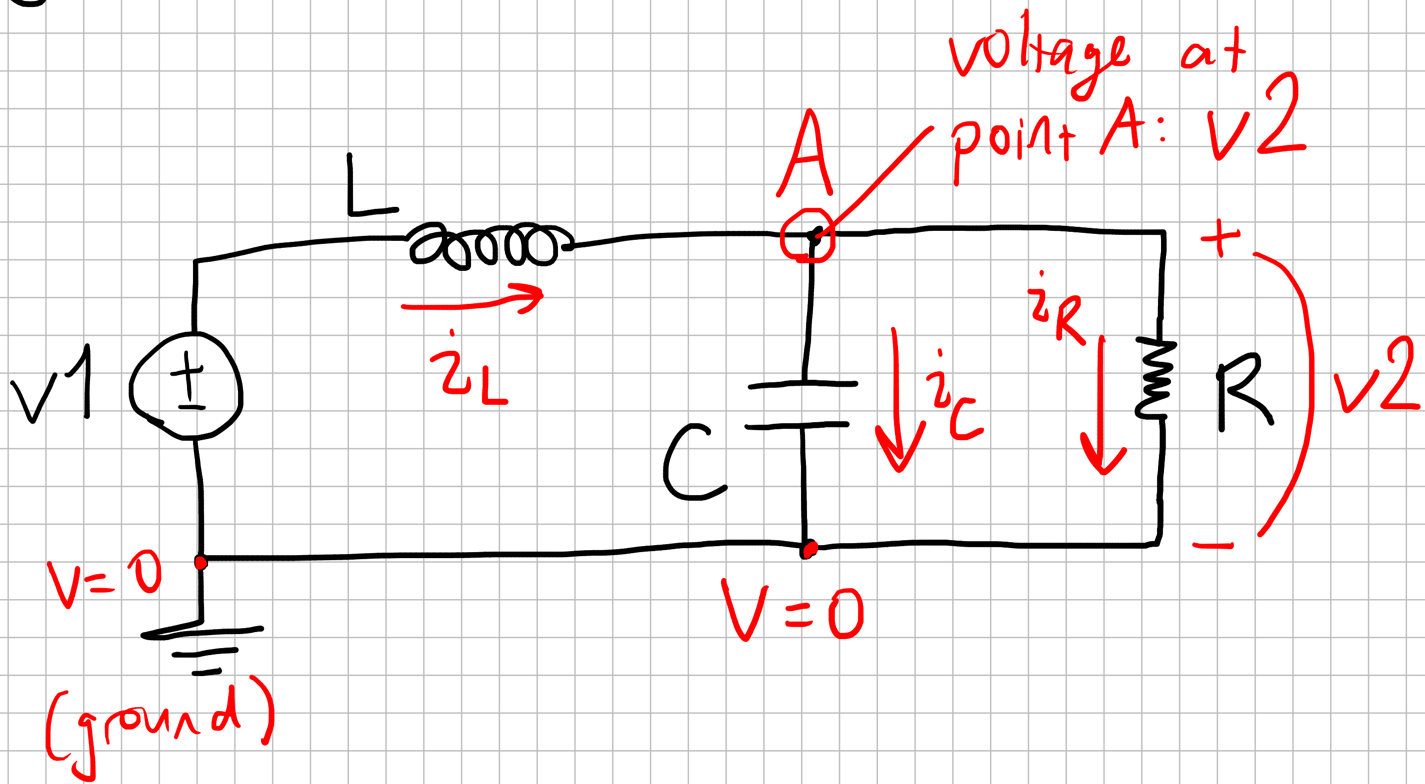
Rewriting KVL with these, we find:

$$\boxed{v = R\dot{q} + L \cdot \ddot{q} + \frac{1}{C} q}$$

Question 2: Consider the schematic of an electrical circuit depicted below, consisting of a resistor, an inductor, and a capacitor (with parameters R, L, and C, respectively). The voltage drop across the resistor is denoted as v2(t). An external voltage is applied to the circuit, denoted as v1(t). Find the differential equation model of the system relating v1(t) and v2(t).



Solution: Define the following symbols for the circuit:



Invoking KCL at point A:

$$i_L = i_c + i_R$$

Invoking KVL for left loop:

$$v1 = V_L + v2$$

The question asks us to relate v1 and v2, thus we need to rewrite V_L and V_c in terms of v2.

Writing the linear model of the inductor: $V_L = L \cdot \dot{i}_L$

Rewriting KVL with this:

$$v1 = L \cdot \dot{i}_L + v2$$

We still need to rewrite $\dot{i}_L$ in terms of v2.

We found KCL for point A as follows: $i_L = i_C + i_R$

Differentiating both sides, we get: $\dot{i}_L = \dot{i}_C + \dot{i}_R$

From linear models of the capacitor and resistor, we can write:

$$i_C = C \cdot \dot{v}2 \qquad i_R = \frac{1}{R} v2$$

Differentiating both:

$$\dot{i}_C = C \cdot \ddot{v}2 \qquad \dot{i}_R = \frac{1}{R} \cdot \dot{v}2$$

Rewriting $\dot{i}_L = \dot{i}_C + \dot{i}_R$

with these, we have:

$$\dot{i}_L = C \cdot \ddot{v}2 + \frac{1}{R} \cdot \dot{v}2$$

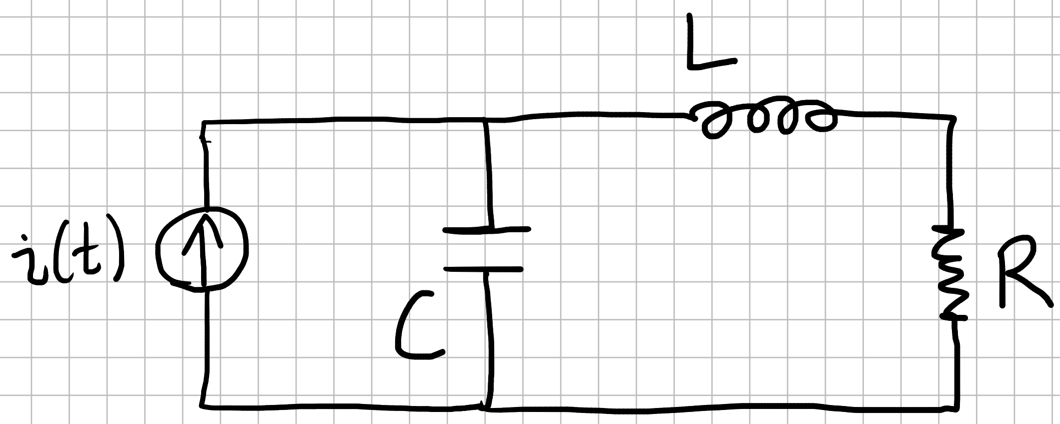
Substituting this in the KVL:

$$v1 = L \cdot \dot{i}_L + v2$$

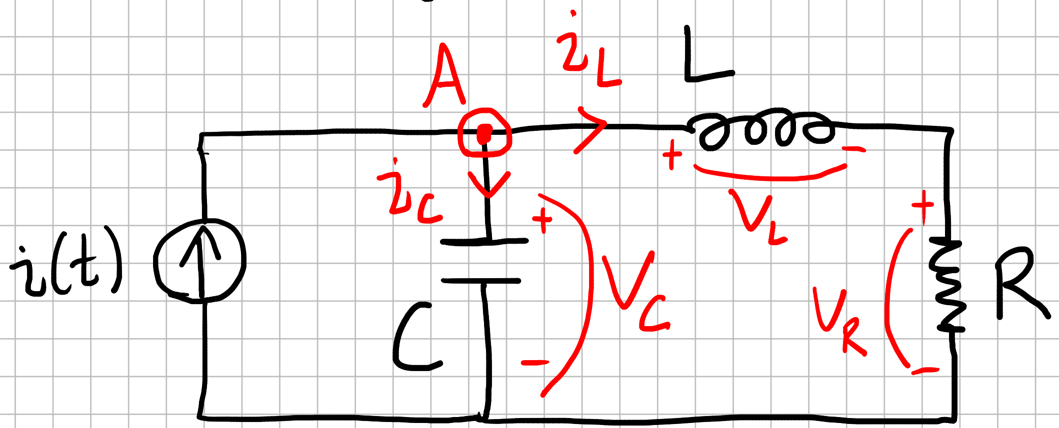
$$v1 = L \cdot \left(C \cdot \ddot{v}2 + \frac{1}{R} \dot{v}2 \right) + v2$$

$$\boxed{v1 = L \cdot C \cdot \ddot{v}2 + \frac{L}{R} \cdot \dot{v}2 + v2}$$

Question 3: Consider the schematic of an electrical circuit depicted below, consisting of a resistor, an inductor, and a capacitor (with parameters R, L, and C, respectively). The current flowing through the inductor is denoted as iL(t). An external current is supplied to the circuit, denoted as i(t). Find the differential equation model of the system relating i(t) and iL(t).



Solution: Define the following symbols for the circuit:



Invoking KCL at point A:

$$i = i_C + i_L$$

Invoking KVL for right loop:

$$V_C = V_L + V_R$$

From linear models of the inductor and the resistor, we can write:

$$V_L = L \cdot \dot{i}_L \qquad V_R = R \cdot i_L$$

Rewriting KVL with these:

$$V_C = L \cdot \dot{i}_L + R \cdot i_L$$

Differentiating both sides:

$$\dot{V}_C = L \cdot \ddot{i}_L + R \cdot \dot{i}_L$$

Multiplying both sides by C:

$$C \cdot \dot{V}_C = L \cdot C \cdot \ddot{i}_L + R \cdot C \cdot \dot{i}_L$$

From linear model of the capacitor: $i_C = C \cdot \dot{V}_C$

This equation thus becomes:

$$i_C = L \cdot C \cdot \ddot{i}_L + R \cdot C \cdot \dot{i}_L$$

From KCL at point A we found:

$$i = i_C + i_L \text{, thus: } i_C = i - i_L$$

Substituting i_C, this equation becomes:

$$i - i_L = L \cdot C \cdot \ddot{i}_L + R \cdot C \cdot \dot{i}_L$$

rearranging to have i_L terms on LHS:

$$\boxed{LC\,\ddot{i}_L + RC\,\dot{i}_L + i_L = i}$$

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Part 2: Mechanical systems

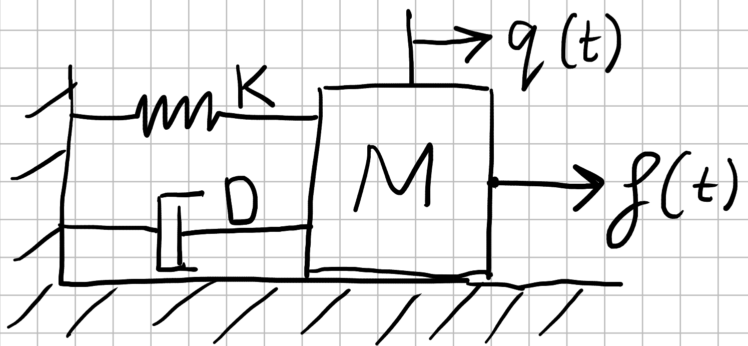
Abbreviations:

FBD: free body diagram

N2L: Newton's second law of motion

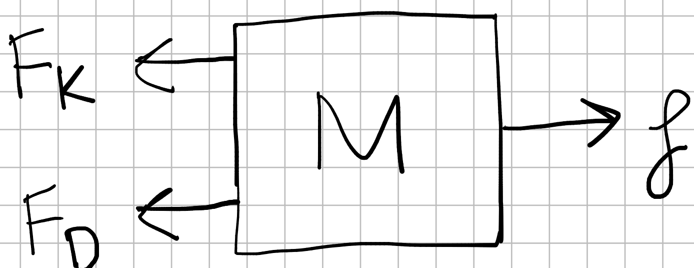
LHS: left-hand side (of the equation)

Question 1: Consider the schematic of a translational mechanical system depicted below, consisting of a mass, a spring, and a damper (with parameters M, K, and D, respectively). Position of the mass is denoted as q(t). An external force is being applied to the system, denoted as f(t). There are no other forces acting on the system.
Find the differential equation model of the system relating f(t) and q(t).



Solution:

Draw the FBD for the mass:



From linear models of spring & damper, we can write:

$$F_K = K \cdot q$$

$$F_D = D \cdot \dot{q}$$

Writing N2L for the mass:

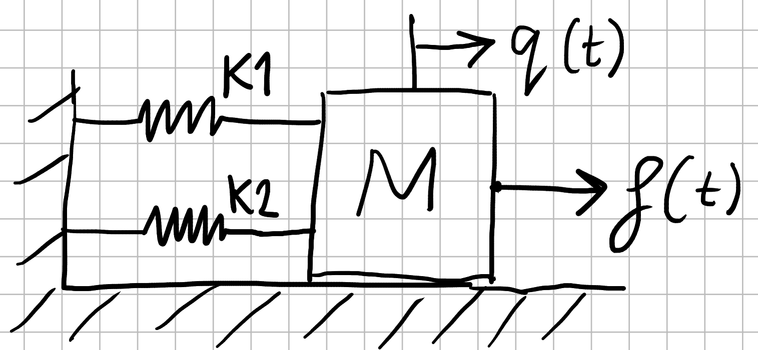
$$\sum_i F_i = Ma = M\ddot{q}$$

$$f - K \cdot q - D \cdot \dot{q} = M\ddot{q}$$

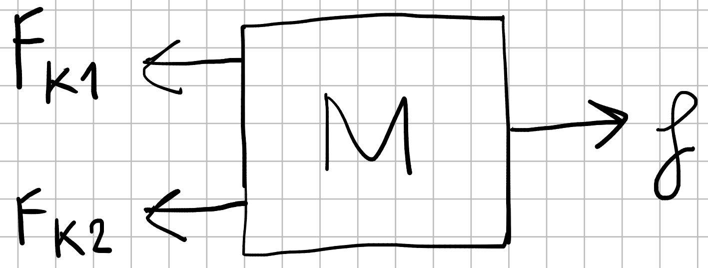
or (rearranging to have q terms on LHS):

$$\boxed{M \cdot \ddot{q} + D \cdot \dot{q} + Kq = f}$$

Question 2: Consider the schematic of a translational mechanical system depicted below, consisting of a mass and two springs (with parameters M, K1, and K2, respectively). Position of the mass is denoted as q(t). An external force is being applied to the system, denoted as f(t). There are no other forces acting on the system. Find the differential equation model of the system relating f(t) and q(t).



Solution: Draw FBD of the mass



From linear model of spring we can write:

$$F_{K1} = K1 \cdot q \qquad F_{K2} = K2 \cdot q$$

Writing N2L for the mass:

$$\sum_i F_i = Ma = M\ddot{q}$$

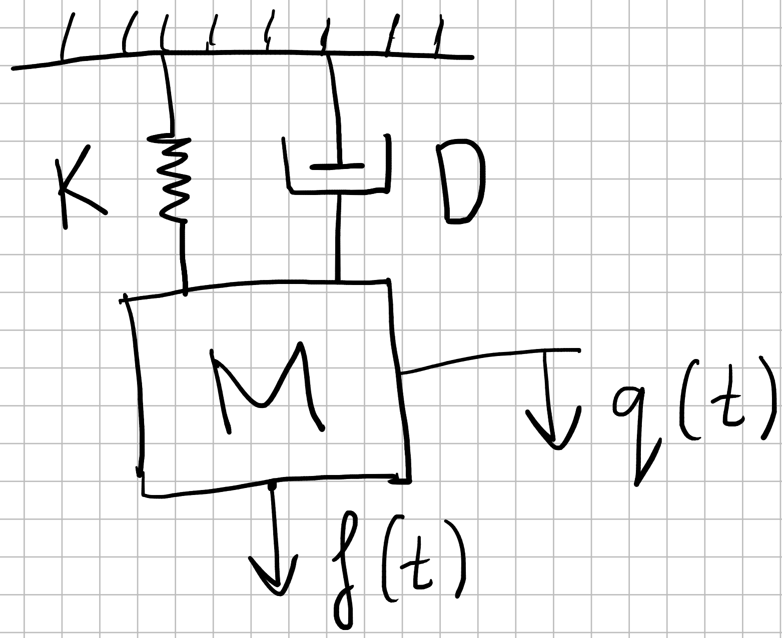
$$f - K1 \cdot q - K2 \cdot q = M \cdot \ddot{q}$$

$$f - (K1 + K2) \cdot q = M \cdot \ddot{q}$$

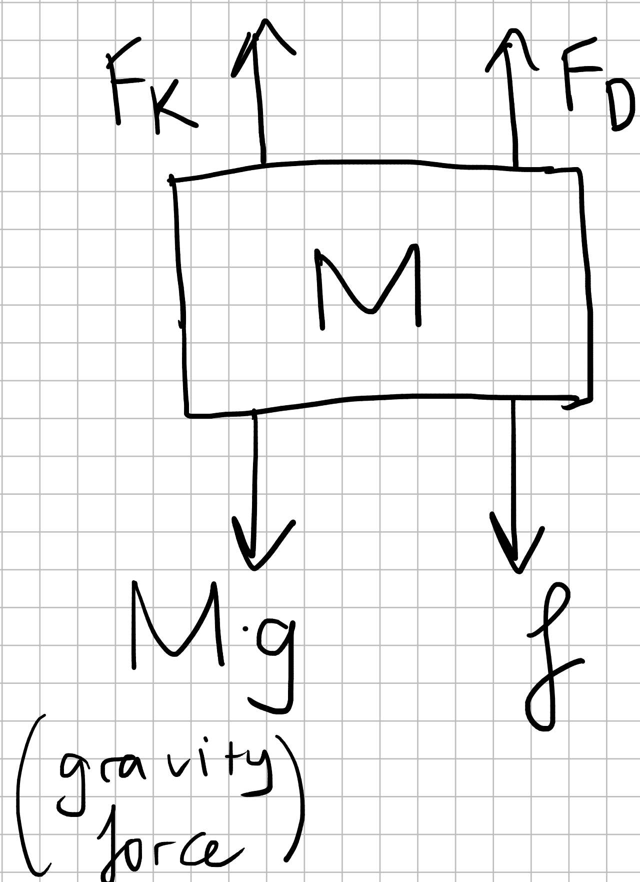
or (rearranging to have q terms on LHS):

$$\boxed{M \cdot \ddot{q} + (K1 + K2) \cdot q = f}$$

Question 3: Consider the schematic of a translational mechanical system depicted below, consisting of a mass, a spring, and a damper (with parameters M, K, and D, respectively). Position of the mass is denoted as q(t). An external force is being applied to the system, denoted as f(t). Gravity is acting on the system, with gravitational acceleration constant g.
There are no other forces acting on the system.
Find the differential equation model of the system relating f(t) and q(t).



Solution: Draw FBD for the mass



From linear models of spring & damper, we can write:

$$F_K = K \cdot q$$

$$F_D = D \cdot \dot{q}$$

Writing N2L for the mass:

$$\sum_i F_i = M a = M \ddot{q}$$

$$f + Mg - K \cdot q - D \cdot \dot{q} = M \ddot{q}$$

or (rearranging to have q terms on LHS):

$$\boxed{M \cdot \ddot{q} + D \cdot \dot{q} + K q = f + Mg}$$

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Exam-style questions with solutions

Part 3: Electromechanical systems

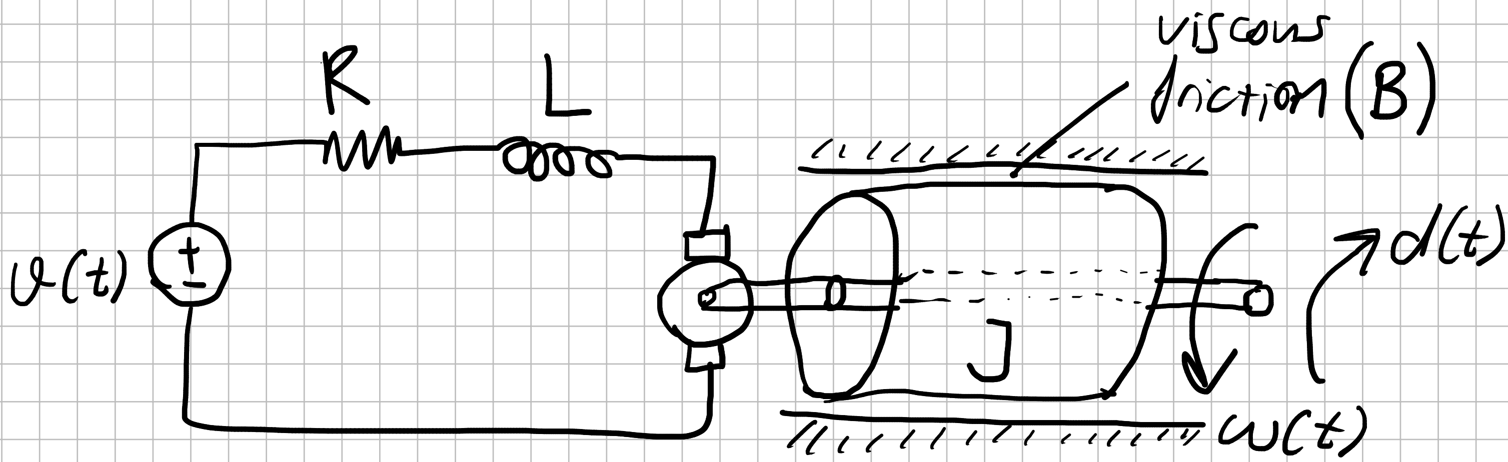
Abbreviations:

emf: electromotive force

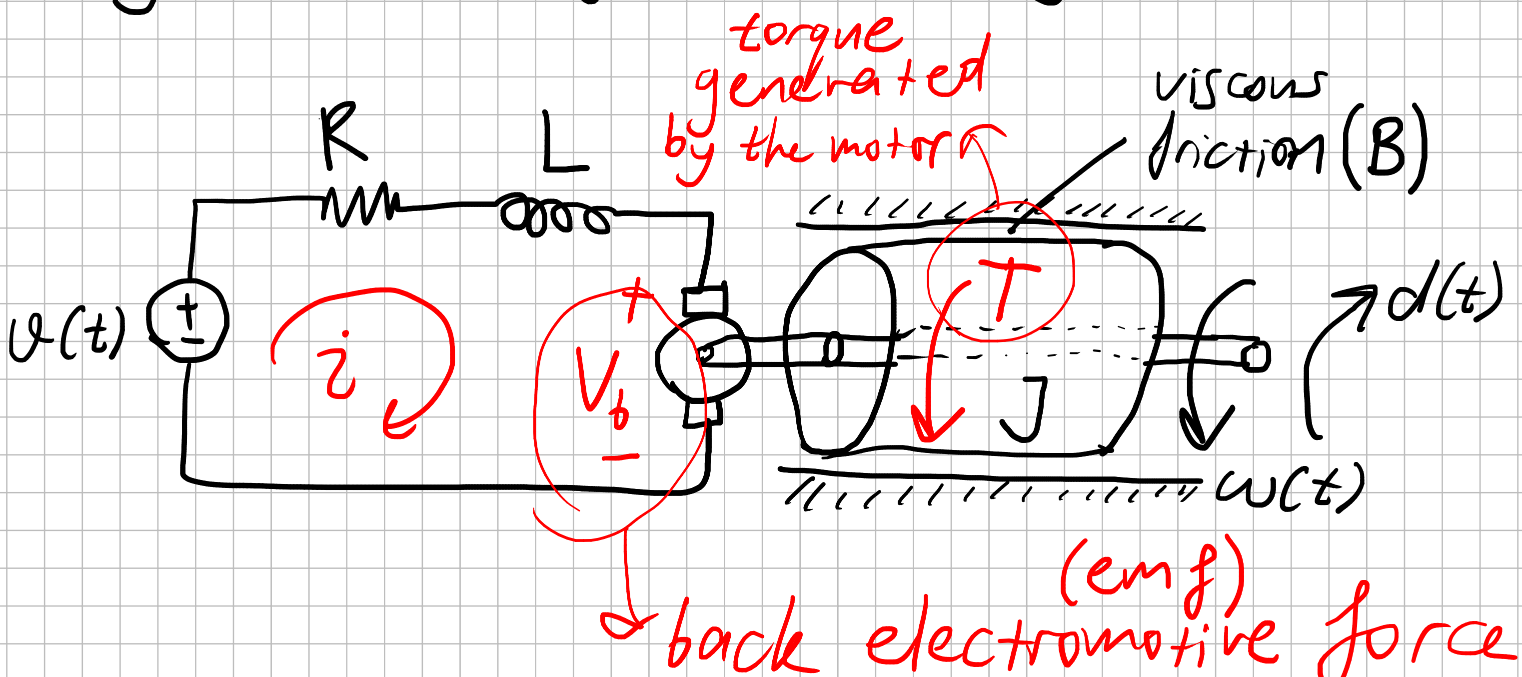
KVL: Kirchhoff's voltage law

N2L: Newton's second law of motion

Question 1: Consider the schematic of an rotational electromechanical system (DC motor) depicted below, consisting of a resistor and an inductor (with parameters R and L) for the electrical part, and a rotating object with a moment of inertia of J and a viscous friction element with constant B, for the mechanical part. The motor torque constant Km is equal to the back electromotive force constant Kb, and is given as K (that is, K = Km = Kb). Angular velocity of the mass is denoted as ω(t). An external voltage is being applied to the system, denoted as v(t). An external load torque is being applied to the system, denoted as d(t). There are no other forces or torques acting on the system. Find the differential equation model of the system relating v(t), d(t) and ω(t).



Solution: Define the following symbols for the system:



Note that T and ω have the same direction, while d is in the opposite direction.

↳ We should notice this from the system schematic given in the question.

Writing KVL for the electrical part:

$$v = V_R + V_L + V_b$$

From linear models of R and L elements:

$$V_R = R \cdot i \qquad V_L = L \cdot \dot{i}$$

Thus, KVL becomes:

$$v = R \cdot i + L \cdot \dot{i} + V_b$$

Writing N2L for the rotational motion of the mechanical part:

$$\sum_i \tau = J \cdot \dot{\omega}$$

$$T - d - B \cdot \omega = J \cdot \dot{\omega}$$

torque generated by the motor

external load torque

torque due to viscous friction

The torque-current relationship is as follows: $T = K \cdot i$

The back emf-speed relationship is as follows: $V_b = K \cdot \omega$

Substituting these into KVL and N2L:

$$v = R \cdot i + L \cdot \dot{i} + K \cdot \omega$$

$$J\dot{\omega} + B\omega - K \cdot i = -d$$

Rewriting this for i alone:

$$i = \frac{J\dot{\omega} + B\omega + d}{K}$$

Differentiating both sides:

$$\dot{i} = \frac{J\ddot{\omega} + B\dot{\omega} + \dot{d}}{K}$$

Substituting these in KVL:

$$v = \frac{RJ}{K}\dot{\omega} + \frac{RB}{K}\omega + \frac{R}{K}d + \ldots$$

$$+ \frac{LJ}{K}\ddot{\omega} + \frac{LB}{K}\dot{\omega} + \frac{L}{K}\dot{d} + K\omega$$

Simplifying by gathering ω and $\dot{\omega}$ terms together:

$$\frac{LJ}{K}\ddot{\omega} + \frac{RJ+LB}{K}\dot{\omega} + \left(\frac{RB}{K} + K\right)\omega = \ldots$$

$$\ldots v - \frac{L}{K}\dot{d} - \frac{R}{K}d$$

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Exam-style questions with solutions

Part 4: Laplace transform and transfer functions

Abbreviations:

ODE: ordinary differential equation

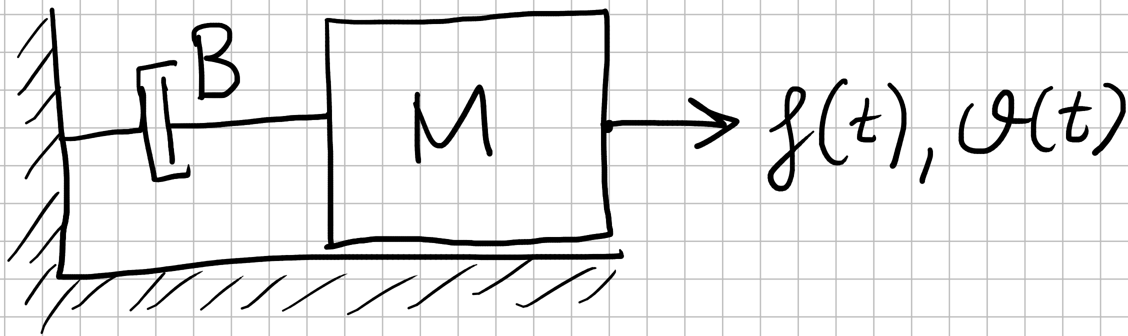
N2L: Newton's second law of motion

LHS: left-hand side (of the equation)

RHS: right-hand side (of the equation)

opamp: operational amplifier

Question 1: Consider the schematic of a translational mechanical system depicted below, consisting of a mass and a damper (with parameters M and B). Speed of the mass is denoted as v(t). An external force is being applied to the system, denoted as f(t). There are no other forces acting on the system. Assuming that the external force is in the form of a step function f(t) = f0*1(t) (with 1(t) denoting the unit step function; and f0 a constant), and that all initial conditions are zero, find the speed of the mass at time equal to t = 5.



Solution: We first need to find the ODE model of the system, and then solve it for $f(t) = f_0 \cdot 1(t)$ to find the solution for $v(5)$.

Writing N2L:

$$\underbrace{f - B \cdot v = M \cdot \dot{v}}$$

this is the ODE model of the system

rearranging to have v terms on LHS:

$$M\dot{v} + Bv = f$$

To solve the ODE (using Laplace transform method), we proceed as follows:

Take Laplace transform of both sides:

$$\mathcal{L}\{M \cdot \dot{v} + Bv\} = \mathcal{L}\{f(t)\}$$

Laplace transform is linear, thus:

$$\mathcal{L}\{M\dot{v}\} + \mathcal{L}\{Bv\} = \mathcal{L}\{f(t)\}$$

$$M \cdot \mathcal{L}\{\dot{v}\} + B \cdot \mathcal{L}\{v\} = \mathcal{L}\{f(t)\}$$

We define the following symbols: (following standard notational convention of using uppercase letters for Laplace transforms of general signals)

$$\mathcal{L}\{v(t)\} = V(s)$$

$$\mathcal{L}\{f(t)\} = F(s)$$

Furthermore, from the Laplace transforms table, we know that:

$$\mathcal{L}\{\dot{v}\} = sV(s) - \cancel{v(0)}^{\;\to 0}$$

(all initial conditions are zero!)

$$\mathcal{L}\{\dot{v}\} = sV(s)$$

Thus, we have:

$$M \cdot s\,V(s) + B \cdot V(s) = F(s)$$

$$(M \cdot s + B) \cdot V(s) = F(s)$$

$$V(s) = \frac{1}{Ms + B} \cdot F(s)$$

We found the relationship between $f(t)$ and $v(t)$ in the Laplace domain. Now we need to substitute for the specific $f(t)$ given in the question and then take the inverse Laplace transform to find the solution to the ODE.

In the question it says:

$$f(t) = f_0 \cdot 1(t)$$

Taking the Laplace transform:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{f_0 \cdot 1(t)\}$$
$$= f_0 \cdot \mathcal{L}\{1(t)\}$$

From the Laplace transform table, we know that: $\mathcal{L}\{1(t)\} = \frac{1}{s}$

Thus, for $f(t)$, we have:

$$\mathcal{L}\{f(t)\} = F(s) = f_0 \cdot \frac{1}{s}$$

Substituting this in the following

$$V(s) = \frac{1}{Ms + B} \cdot F(s)$$

$$V(s) = \frac{1}{Ms + B} \cdot f_0 \cdot \frac{1}{s}$$

Rewriting to have 1 as the coefficient of the s term in the denominator:

$$V(s) = \frac{f_0}{M} \cdot \frac{1}{s \cdot (s + B/M)}$$

Now we need to take the inverse Laplace transform of $V(s)$ to obtain the ODE's solution $v(t)$.

From the Laplace transform table, we know that:

$$\mathcal{L}\left\{\frac{1}{a}\cdot(1-e^{-at})\right\} = \frac{1}{s\cdot(s+a)} \quad \left(a \in \mathbb{R}, \text{ constant}\right)$$

To see this, use $\mathcal{L}\left\{\frac{1}{a}\cdot 1(t)\right\} = \frac{1}{a}\cdot\frac{1}{s}$ and $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$ together, as follows:

$$\frac{1}{a}(1-e^{-at}) = \frac{1}{a} - \frac{1}{a}e^{-at}$$

Taking Laplace transform of RHS

$$\mathcal{L}\left\{\frac{1}{a}\right\} = \frac{1}{a}\cdot\frac{1}{s} \quad \left(\text{since } \mathcal{L}\{1\} = \frac{1}{s}\right)$$

$$\mathcal{L}\left\{\frac{1}{a}\cdot e^{-at}\right\} = \frac{1}{a}\cdot\frac{1}{s+a} \quad \left(\text{since } \mathcal{L}\{e^{at}\} = \frac{1}{s-a}\right)$$

combining the two we have:

$$\frac{1}{a}\cdot\frac{1}{s} - \frac{1}{a}\cdot\frac{1}{s+a} = \frac{1}{a}\cdot\left(\frac{1}{s} - \frac{1}{s+a}\right) = \frac{1}{s\cdot(s+a)}$$

Since $\mathcal{L}\left\{\frac{1}{a}\left(1-e^{-at}\right)\right\} = \frac{1}{s(s+a)}$,

the inverse Laplace transform of the RHS of $V(s) = \frac{f_0}{M} \cdot \frac{1}{s \cdot (s + B/M)}$

is as follows:

$$v(t) = \frac{f_0}{\cancel{M}} \cdot \frac{\cancel{M}}{B} \cdot \left(1 - e^{-B \cdot t/M}\right)$$

$$v(t) = \frac{f_0}{B}\left(1 - e^{-Bt/M}\right)$$

which is the solution of the ODE.

Evaluating this at time $t = 5$,

we find:

$$\boxed{v(5) = \frac{f_0}{B}\left(1 - e^{-5B/M}\right)}$$

Question 2: Consider the electrical circuit from question 1,
Part 1: Electrical systems. Assume that all initial conditions are zero.
Find the transfer function model of the system from v(t) to q(t).

Solution: The ODE model of the system is (see solution of Q1, Part 1):

$$v = R\dot{q} + L\cdot\ddot{q} + \frac{1}{C}q$$

Taking Laplace transforms (considering all initial conditions zero) of all terms, we have:

$$V(s) = L\cdot s^2\cdot Q(s) + R\cdot s\cdot Q(s) + \frac{1}{C}Q(s)$$

$$V(s) = Q(s)\cdot\left(Ls^2 + Rs + \frac{1}{C}\right)$$

The transfer function model for this system is the ratio of the Laplace transforms of q(t) (the output) and v(t) (the input). Thus:

$$\frac{Q(s)}{V(s)} = \boxed{\frac{1}{Ls^2 + Rs + 1/C}}$$

Question 3: Consider the mechanical system from question 1, Part 2: Mechanical systems. Assume that all initial conditions are zero. Find the transfer function model of the system from f(t) to q(t).

Solution: The ODE model of the system is (see solution of Q1, Part 2):

$$M \cdot \ddot{q} + D \cdot \dot{q} + Kq = f$$

Taking Laplace transforms (considering all initial conditions zero) of all terms, we have:

$$Ms^2 Q(s) + Ds\,Q(s) + K\,Q(s) = F(s)$$

$$Q(s) \cdot (Ms^2 + Ds + K) = F(s)$$

$$\frac{Q(s)}{F(s)} = \boxed{\frac{1}{Ms^2 + Ds + K}}$$

Question 4: Consider the electromechanical system from question 1, Part 3: Electromechanical systems. Assume that: 1) L and B parameters are equal to zero. 2) External load torque d(t) is equal to zero. 3) All initial conditions are zero.
Find the transfer function model of the system from v(t) to ω(t).

Solution: The ODE model of the system is (see solution of Q1, Part 3)

$$\frac{LJ}{K}\ddot{\omega} + \frac{RJ+LB}{K}\dot{\omega} + \left(\frac{RB}{K} + K\right)\omega = \dots$$

$$\dots v - \frac{L}{K}\dot{d} - \frac{R}{K}d$$

Using the assumptions $L=0$, $B=0$, and $d(t)=0$, the ODE model simplifies to:

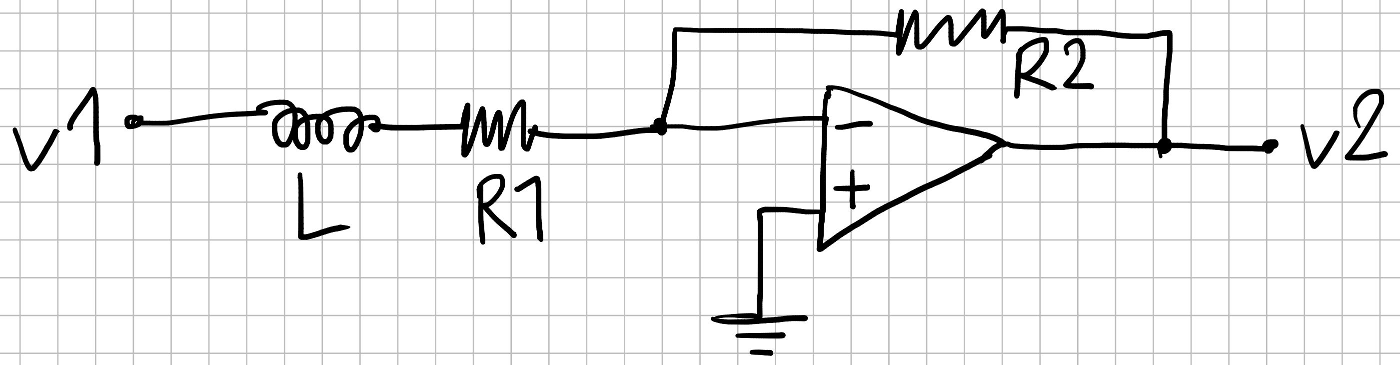
$$\frac{RJ}{K}\dot{\omega} + K\omega = v$$

Taking Laplace transforms (considering all initial conditions zero) of all terms, we have:

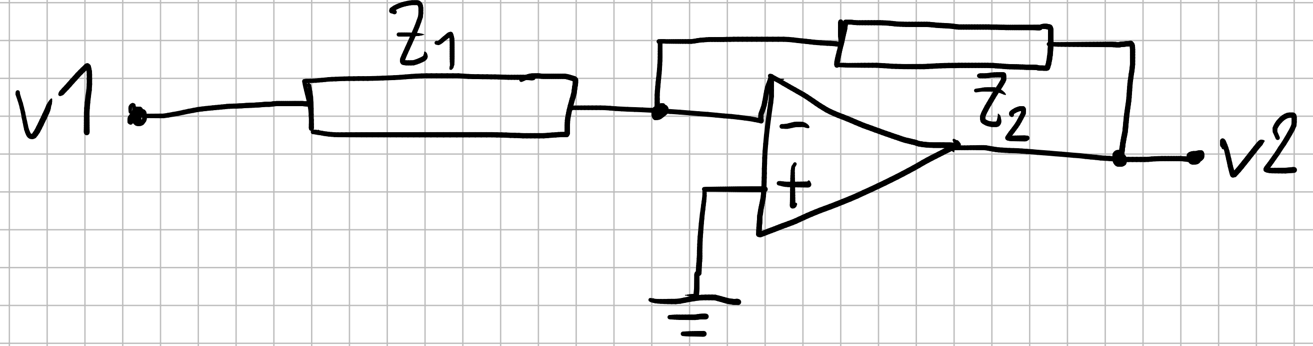
$$\frac{RJ}{K} s\,\Omega(s) + K\cdot\Omega(s) = V(s)$$

$$\frac{\Omega(s)}{V(s)} = \boxed{\frac{1}{\frac{RJ}{K}\cdot s + K}}$$

Question 5: Consider the schematic of an electronic circuit depicted below, consisting of an ideal opamp, together with two resistors and an inductor (with parameters R1, R2, and L, respectively).
The voltages at the two ends of the circuit are denoted as v1(t) and v2(t).
Find the transfer function model of the system from v1(t) to v2(t).



Solution: From the general form of the inverting opamp circuit, we know:



$$\frac{V2}{V1} = -\frac{Z_2}{Z_1} \quad \left(\text{transfer function}\right)$$

Specifically for the circuit in the question, we have: $Z_1 = Ls + R1$, $Z_2 = R2$

Thus: $$\frac{V2}{V1} = \boxed{\frac{-R2}{Ls + R1}}$$